On the determination of the Average Risk attaching to the grant of Insurances upon Lives. By Dr. M. Kanner, *Actuary of the Frankfort Life Assurance Company.*

[Translated from the *Deutsche Versicherungs-Zeitung.*]*

AN Insurance Office, which in consideration of fixed premiums undertakes the payment of a sum on the death of the assured, strictly speaking lays a wager with each of the assured, in which the stakes are proportional to the probabilities of the happening of the two contrary events—his living and his dying. Every assurance of a sum payable at death admits of being divided into partial assurances for a fixed period, as a year. Let the sum assured be S and the probability of death within a year p, then pS will be the stake of the assured, or the premium; and $(1-p)$S the stake of the office, *i.e.* the amount which the office would lose in case of the death of the assured within the year. The two stakes together therefore amount to the sum assured.

If the number of the assured be sufficiently large, it may happen that the office on the whole neither gains nor loses. Of all the cases that may happen, this is always the most probable, if all the lives are insured for the same amount and have the same probability of death. Moreover, the probability of this case increases with the number of the assured, so that it approaches without limit to certainty, and would reach certainty if the number of the assured could become infinitely great. This proposition is based on the Principle of Large Numbers, of which the celebrated mathematician, James Bernoulli, succeeded in obtaining a strict demonstration after twenty years' reflection; and which then ceased to be a mere result of observation, and was acknowledged as a mathematical truth.

The case in which the office neither gains nor loses is only physically possible in the rare event when, the sums assured being all equal, and n in number, pn is an integer; or when, the sums assured being different, some combination of the deaths is possible, such that the sum of the claims to be paid is equal to the sum of the premiums. The greater the number of the assured, and the more nearly the sums assured approach equality, the greater also will be the probability of there being very little gain or loss; and therefore it is evident that the risk of a deviation from the most probable case becomes greater as the numbers are smaller, and the sums assured more unequal.

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Insurance Companies calculate their rates on the supposition that the most probable case will happen; so that their profit must consist only of the excess of interest realized beyond that calculated upon, and of a portion of the loading, or addition to the net premium. So also, the Premium-Reserve is formed by the accumulation at interest of the balance of the premiums which would remain at the time of valuation after payment of the claims, if the most probable case had constantly happened. This Reserve again has this further characteristic, that on the supposition of the most probable case happening for the future, it will exactly suffice for the discharge of the liabilities undertaken by the office.

From this it results that the Premium-Reserve by itself does not offer the guarantee that may reasonably be desired; since the possibility of a loss is not provided for. For this purpose, therefore, every Insurance Office ought, year by year, to have a separate Fund (Capital-Reserve), the magnitude of which is to be determined at the beginning of each year according to the *amount of the insurances in force.** To determine what this Fund should be, is a problem in probabilities; for we cannot think of providing for all possible risks to their full value, but we may calculate the *average risk*, by taking into account every possible case according to its probability.

From the foregoing explanation of the nature of the risk, it is evident that this is still the case if the fundamental Mortality Table is perfectly accurate, *i.e.*, if it represents the actual probabilities of death. But absolute accuracy is never to be obtained, and this circumstance is a fresh danger for the Insurance Office, which however is met by the addition made to the net premiums.

It is not my present purpose to discuss the theory of the construction of Tables of Mortality, and I will now only point out that a Table, founded upon the arithmetic mean of a series of observations free from error, made independently of each other, and under similar conditions, indicates the most probable values of the actual probabilities of death, whilst these last must necessarily remain unknown.

Suppose that we have for a given age the following data:—

Out of n_1 living, m_1 have died within a year

„ n_2 „ m_2 „ „

$\vdots$ $\vdots$

„ n_k „ m_k „ „

* The word in the original, here and elsewhere, is *Versicherungsbestand*, which the author states would be expressed in French by *état des assurances.*

Then the greatest probability is to be ascribed to that value of the unknown probability of death, by the adoption of which the concurrence of the observed results has the greatest probability.

Let x be any probability of death between 0 and 1, and for brevity put

$$n_1 + n_2 + \ldots + n_k = \Sigma(n)$$
$$m_1 + m_2 + \ldots + m_k = \Sigma(m)$$

Then the probability of any one of the above results $(m.n)$, (that of n living, m die within a year) will be expressed by

$$B.x^m(1-x)^{n-m}$$

where B denotes the mth coefficient of the expansion of $(1+x)^n$ $\left[=\frac{\lfloor n}{\lfloor m \lfloor n-m}\right]$. The probability of the happening of all the k independent events is equal to the product of their separate probabilities, *i.e.* to

$$C.x^{\Sigma(m)}(1-x)^{\Sigma(n)-\Sigma(m)}$$

where C denotes the product of all the k Binomial coefficients. This product is to be a maximum with regard to x. Hence putting the first differential coefficient equal to 0, we obtain, (omitting the constant factor C)

$$\Sigma(m)x^{\Sigma(m)-1}(1-x)^{\Sigma(n)-\Sigma(m)} - \{\Sigma(n)-\Sigma(m)\}x^{\Sigma(m)}(1-x)^{\Sigma(n)-\Sigma(m)-1} = 0$$

whence $$x = \frac{\Sigma(m)}{\Sigma(n)}.$$

We can easily prove that for this value, the second differential coefficient is negative, and therefore a maximum occurs; and thus is established the proposition, that the arithmetical mean represents the most probable value of the probability of death.

From the foregoing it evidently follows that there always exists a risk for the office independently of the unavoidable errors in the construction of the Table of Mortality.

Now in order to determine the risk for a given amount of existing insurances, and for the period of one year, I start from the simple principle, that the premium which the assuring office would have to pay to another, in order to insure itself against every possible loss during the year, furnishes a measure of the risk, or represents *the average risk,* an expression which will be justified further on.

If we wished to calculate this premium directly, we must take into consideration, all the 2^n possible cases which may arise among n assured persons, and multiply the loss in each case nto the

probability of its occurrence; and the sum of all these products would represent the premium sought. But we see that such a process would be extremely laborious, and that with only a moderate number of lives insured many generations would be required for performing the calculations. But there fortunately exists another method by which we can arrive at the result without trouble. This is by calculating, not the mathematical expectation of *loss*, but that of *gain*, these two being, as will be shown directly, always equal to each other.

The loss in consequence of the death of a person of the age x insured for S, would consist of S diminished by the last reserve, R_x, and the last premium, ϖ_x, and a year's interest upon both of these, *i.e.*

$$S-(R_x+\varpi_x)(1+i).$$

But there exists between the old Reserve, R_x, and the new, R_{x+1}, the following easily proved relation

$$(R_x+\varpi_x)(1+i)=R_{x+1}+p(S-R_{x+1}),$$

where p denotes the probability of dying in the $(x+1)$th year. We can therefore bring the actual loss into the following form,

$$S-R_{x+1}-p(S-R_{x+1}).$$

If now we call $S-R_{x+1}$ the *reduced* sum assured, and $p(S-R_{x+1})$ the *reduced* premium, we can enunciate in the following terms the theorem mentioned above. *If there are n persons of the same age, whose probability of dying in a year is p, and whose reduced sums assured are $a_1, a_2, \ldots a_n$, the sum of the mathematical expectations of gain is equal to the sum of the mathematical expectations of loss.*

Proof.—The probability that m specified persons insured for the reduced sums $a_p, a_q, a_r, \ldots$ will die, and the remaining, $n-m$, will live, is represented generally by $p^m(1-p)^{n-m}$. If we consider loss as negative gain, we can represent the gain or loss, whichever it may be, in this case by

$$p\Sigma(a)-(a_p+a_q+\ldots\ldots+a_t) \quad . \quad . \quad . \quad (1)$$

where for brevity, we put

$$a_1+a_2+\ldots+a_n=\Sigma(a)$$

and where $p\Sigma(a)$ represents nothing else than the expected loss of the office. The mathematical expectation of gain or loss for the case supposed is therefore generally

$$p^m(1-p)^{n-m}\{p\Sigma(a)-(a_p+a_q+\ldots+a_t)\} \quad . \quad . \quad (2)$$

and we shall get the several mathematical expectations for all possible cases, by giving m in the above expression all integral values from 0 to n in succession, and then for each value of m forming all the combinations of the quantities $a_1, a_2, a_3, \ldots\ldots a_n$, taken m at a time. The number of all the cases is equal to the sum of all the coefficients of $(1+x)^n$, that is, to 2^n.

In order to find the value of all the gains or losses that happen for a given value of m, we have to find the value of the expression (1) for each of the possible $\frac{n(n-1)(n-2)\ldots}{\underline{|m}}$ cases, and to take the sum of these values. We get as the positive part of that sum

$$\frac{n(n-1)\ldots\ldots}{\underline{|m}}p\Sigma(a).$$

The negative part consists of the sum of the several terms of all the combinations, m at a time, of the quantities $a_1, a_2, \ldots a_n$. In that sum, every quantity must occur $\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}$ times; for if we consider those combinations in which a_p occurs, we see that they are so formed that a_p occurs along with all the possible combinations, $(m-1)$ at a time, of the remaining $(n-1)$ quantities, so that the number of the combinations in which a_p occurs, is

$$\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}.$$

Since this is true for each of the m quantities, we obtain as the negative part of the sum,

$$\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}(a_1+a_2+\ \ldots\ +a_n)=\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}\Sigma(a).$$

Combining this with the positive part as found above, we get the sum equal to

$$\Sigma(a)\left\{\frac{n(n-1)\ldots}{\underline{|m}}p-\frac{(n-1)(n-2)\ldots}{\underline{|m-1}}\right\}.$$

This expression may also be written

$$\frac{\Sigma(a)}{n}\left\{\frac{n(n-1)\ldots}{\underline{|m}}np-\frac{n(n-1)\ldots}{\underline{|m-1}}\right\}=\frac{\Sigma(a)}{n}.\frac{n(n-1)\ldots}{\underline{|m}}(np-m)$$

The sum of all the mathematical expectations of gain or loss for a given value of m is therefore

$$\frac{\Sigma(a)}{n}.\frac{n(n-1)\ldots}{\underline{|m}}(np-m)p^m(1-p)^{n-m}.$$

Giving m now every value from 0 to n, we finally obtain as the sum of the mathematical expectations for all the 2^n cases,

$$\frac{\Sigma(a)}{n}\left[np(1-p)^n+(np-1)n(1-p)^{n-1}+\ldots+(np-n)p^n\right]$$

The expression in the square brackets vanishes identically; for we can also write it in the following form

$$np\{(1-p)^n+n(1-p)^{n-1}p+\ldots+p^n\}$$
$$-np\{(1-p)^{n-1}+(n-1)(1-p)^{n-2}p+\ldots+p^{n-1}\}$$

and it is evident, that

$$(1-p)^n+n(1-p)^{n-1}p+\ldots+p^n=\{(1-p)+p\}^n=1$$

and $(1-p)^{n-1}+(n-1)(1-p)^{n-2}p+\ldots+p^{n-1}=\{(1-p)+p\}^{n-1}=1$

so that the expression in the square brackets reduces to $np-np=0$. But if the whole expression vanishes, the sum of the positive terms must be equal to the sum of the negative; *i.e.* the sums of the mathematical expectations of gain and loss are equal to each other.

We might give the proposition just proved the greatest generality of which it admits, and prove that

For any amount of insurances, whatever may be the ages of the lives insured, and whatever the nature of the insurances, the sums of the mathematical expectations of gain and loss for any interval of time, are equal to each other, if only the premiums are calculated upon the supposition of the most probable case.

Though we do not here require that greater generality, yet I venture to think that in the foregoing demonstration I have pointed out the way to generalization.

It may now be useful to illustrate the method of calculation by two examples.

1. *Problem.*—Let it be required to determine the average risk during a year, when 15 persons of the same age are insured, each for the reduced sum of 5000 thalers, the probability of death within a year being ·02.

Solution.—The sum of the reduced premiums $p\Sigma(a)$, which occurs in formula (2) is here equal to $\cdot 02\times 15\times 5000=1500$, which amount the office would expect to lose in the most probable case. Whatever beyond this amount it might pay, would be a *loss* in the sense in which we have throughout wished the term to be understood.

There are here $2^{15}-1$ cases of loss; because there is only one case in which the office gains, that, namely, in which no

death occurs. The mathematical expectation of gain is therefore $\cdot 98^{15} \times 1500 = 1108$, which value represents at the same time the average risk.

2. *Problem.*—Let the following insurances be given with the common probability of death ·03.

	No. of Persons insured.		Reduced Sums assured.	Reduced Premiums.
A	1	insured for 5000 each	5000 Thlr.	150 Thlr.
B	6	„ 3000 „	18000 „	540 „
C	10	„ 1000 „	10000 „	300 „
D	30	„ 400 „	12000 „	360 „
Total	47		45,000 Thlr.	1,350 Thlr.

What is the average risk within a year?

Solution.—The cases of gain are as follows:

(1)	There may die	0 person	——	in which case there is a gain 1350
(2)	„	1 „	out of C	„ „ „ 350
(3)	„	1 „	out of D	„ „ „ 950
(4)	„	2 persons	out of D	„ „ „ 550
(5)	„	3 „	out of D	„ „ „ 150

If these cases are separately analyzed and their probabilities determined, we shall find as the mathematical expectation of gain,

$$
\begin{aligned}
97^{47} \times 1350 &= 322\cdot 56 \\
10 \times \cdot 03 \times \cdot 97^{46} \times 350 &= 25\cdot 86 \\
30 \times \cdot 03 \times \cdot 97^{46} \times 950 &= 210\cdot 60 \\
\frac{30.29}{2} \times \cdot 03^{2} \times \cdot 97^{45} \times 550 &= 54\cdot 68 \\
\frac{30.29.28}{2\cdot 3} \times \cdot 03^{3} \times \cdot 97^{44} \times 150 &= 4\cdot 30 \\
\text{Total} \quad . \quad . \quad & 618\cdot 00
\end{aligned}
$$

The average risk is therefore $= 618$.

If we should have to do with a large number of persons insured for very different sums, the calculations lead, as in the most important applications of the theory of probabilities, to products of a very great number of unequal factors, in which cases we must have recourse to the method of approximation given by Laplace in his great work—*Théorie analytique des probabilités*—which requires for its demonstration the most refined analysis.

The above given method of calculating the average risk extends to all kinds of Life Insurances, as for example, to joint lives and the like. But for Endowments, in which on the contrary the number of the cases of loss is far smaller, we can determine by a direct process the sum of the mathematical expectations of loss.

I now turn to the practical side of the subject, in order to place the application of the mathematical idea of the average risk in a clear light. The opinions of authors in regard to this idea are as wide apart from one another as their theories and the results to which they are led. Is then the question of Risk really of such a vague nature as the various methods of treating it would appear to indicate? From a practical point of view this is undoubtedly the case; but by no means from the mathematical point of view, provided that the question itself is put mathematically.

Thus as long as we simply speak of *a risk*, we have just as little in view a mathematically defined idea as when we speak of the further *duration of life* by itself; but just as this last is only determined by means of its average value, so also we cannot speak of Risk in a mathematical sense, until we introduce its average value. The Risk to an Insurance Office is every disadvantageous possibility of a loss of any amount, and is therefore in itself indeterminate. But if we imagine any amount of existing insurances frequently repeated, so perhaps, that we represent to ourselves an indefinite number of offices all with the same amount of insurances in force, then will the total result in a given time for all the offices taken together, show neither gain nor loss, if their number is assumed infinitely great. The individual offices nevertheless will have to show, some, gains, and others, losses, of various magnitudes; and indeed, all possible cases will appear in proportion to their respective probabilities. The limit of the ratio of the total losses to the number of offices represents the *average* loss, which is equal to the *average* gain; and herein lies the practical meaning of the *average risk*, as well as the justification of the term.

There arises now the important question:—Is this average risk proper to take the place of the Capital-Reserve? Or, in other words, is the office perfectly secure if it possesses such a Fund in addition to its Premium-Reserve?

This question withdraws itself from mathematical treatment because here the question is as to a magnitude which can only be arbitrarily assigned, and for the determination of which, mathematically expressed conditions are not attainable. The question here is regarding a number dependent on opinion, by means of which it is to be specified, from what multiple of the average risk the Capital-Reserve must be formed. This number would determine the degree of security of the office, whilst the average risk itself represents the measure of the danger for different amounts of existing insurances.

The determination of the average risk is of particularly great importance for Mutual Companies. Suppose, for instance, that there are two Mutual Insurance Offices, the one with the amount of insurances assumed in the first problem given above, and the other with the amount assumed in the second problem; then would the two offices offer the same degree of security, if they set aside as Capital-Reserve the same multiple of the respective average risks. If an office should once decide on thus setting aside a fixed multiple, it could then, as the amount of existing insurances altered in the course of time, change its Capital-Reserve also, according to the above rules; and by retaining the same multiple maintain always the same degree of security. If, for example, an office had reserved for the amount of insurances in Problem (1) ten times the average risk, *i.e.* 11,080 Thalers, then if at another time the amount of insurances became such as is supposed in Problem (2), it would consequently have to reduce its Capital-Reserve to 6180 Thalers. By this means the Capital-Reserve might be safely regulated upon a scientific basis, whereas the usual course of reserving a certain part of the profits, makes the stability of the office depend upon the extent of the profits it may chance to realize— a mode of proceeding so evidently absurd, that we cannot but wonder at its being still followed.

I do not consider it necessary to explain further how the calculation must be made when the lives assured are of different ages, and we have therefore to do with different probabilities of death; for, the theory having been laid down, it must be left to the skill of the actuary to find the shortest way for arriving at a satisfactory approximation in any particular case.

It may perhaps be objected to the preceding method of calculation that it determines the average risk only for a given time, whereas the danger continues until all the insurances have run out. Against this it must be remembered that the amount of the existing insurances will be so essentially altered after a short time by the termination of existing policies on the one hand, and the introduction of new policies on the other hand, that it would be but an idle task to carry on the calculation to the extinction of all the insurances. Such a calculation however for single insurances would be of interest for the purpose of enabling us to compare the individual risks that result from the various kinds of insurances, and thus obtain a rule for loading the net premiums in the different tables of rates. This problem also can easily be solved by the application of the principles above set forth; namely,

by multiplying the possible losses into their respective probabilities, and expressing the value of the average risk by the sum of all the products so obtained.

Before proceeding to a review of the various modes of treatment which our subject has received, it appears necessary to give a summary of the preceding arguments with the view of throwing further light on some important points.

In determining the idea to be attached to the word *Risk* we proceeded on the supposition that there exist in the Table of Mortality no errors—whether such as may have occurred in their compilation, or such as are caused by the unknown deviations from the actual probability of death, that is to say, from the true law of mortality, of which we can only seek for and obtain the most probable values. We have seen that even under this supposition we can never with any certainty expect the mortality to agree with the tables: but can only say that the mortality indicated by the tables is of all others the most probable, provided its occurrence is not physically impossible. On this point we often meet with erroneous views. People too often speak of "accidental deviations" as if the mortality under normal conditions (that is to say, so long as no new causes of death supervene) must of necessity, or with a high degree of probability, conform to the Table of Mortality. We take the statements of the Table somewhat too literally if we say "out of so many born, so many must be still alive after a certain time"; for the Table can only indicate the limit of the proportion remaining alive out of an infinitely large number of persons born. If then among a finite number of persons, no matter how large, another proportion appears, there would yet be not the least ground for perceiving any special causes for it; for though out of many possible events there is one which has a greater probability than any other, yet is it very often more probable that some one or other of the remaining events will happen rather than that particular event. For example, if we have 100 persons of the same age with a probability equal to ·02 of dying within a year, then the probability, that of these 100 persons two will die within the first year, according to the Mortality Table, is only ·27; and we can therefore wager 8 to 3 that that event will not happen.

This single example suffices to show how worthless are all those calculations and conclusions given at such length in the reports of Companies, and exhibiting the deviations from the so-called expected mortality, if these have been in existence only a short time with a small number of insurances in force. A Company may rightly

express regret for the unfavourable mortality it has experienced; but it does wrong to attribute its losses to deviations from the *law*, for that very mortality which *has occurred* is itself the *law*.

It is a very interesting question whether there is in nature such a thing as a fixed Law of Mortality; or in other words, whether, as the number of observations increases, the rates of mortality approximate to a fixed value. The results of statistics, which always relate to fixed numbers and fixed conditions, may indeed establish a strong presumption in favour of the existence of such a Law, but cannot demonstrate it by evidence.

It was however reserved to analysis to answer this question. Poisson has demonstrated that the definite integral which represents the average duration of life, preserves a constant value so long as there is no material alteration in any one of the known or unknown causes of death, including, among others, the various constitutions of new born children, the influences of climate, and the diseases and pecuniary circumstances of the inhabitants of the country. Take for instance an infinite number of various constitutions of new born children, for which the infinitely small probabilities of living exactly the time t are respectively

$$p'dt, p''dt, p'''dt, \ldots.$$

and further let

$$\mathrm{P}', \mathrm{P}'', \mathrm{P}''', \ldots.$$

denote the probabilities of these constitutions. If then, for brevity we put

$$p'\mathrm{P}'+p''\mathrm{P}''+p'''\mathrm{P}'''+ \ldots. = \mathrm{Z}$$

Z will represent a definite Integral, and $\mathrm{Z}dt$ express the probability of t which extends to all possible constitutions.

The mathematical expectation of life of a new born child of unknown physical constitution is thus expressed by the definite integral

$$\int_0^\infty \mathrm{Z}tdt,$$

which has a fixed though unknown value.

If s denote the sum of the ages at which a very great number, n, of individuals born in the same country and at the same time have died, then we have very approximately, and with the highest degree of probability,

$$\frac{s}{n}=\int_0^\infty \mathrm{Z}tdt.$$

This is one of the two remarkable equations which include in

general terms the principle of large numbers, as enunciated by Poisson, who has demonstrated it by the help of a masterly analysis.

The principle of large numbers is the only true point of view from which to contemplate the law of mortality; and a law being adopted, all the probabilities of death are also given, to which we must adhere in all our calculations, so that new hypotheses are neither necessary nor admissible.

From this simple consideration the whole Theory of Risk at once follows, as here introduced, inasmuch as we have adopted the premiums for the insurance against every possible loss as a measure of the danger, and have finally found these premiums to be identical with the average loss or average gain. If we choose to consider the deviations of the mortality from the most probable case as errors, then we have the average error together with the given probabilities of that error, exactly as in observations of natural phenomena, involving measurements, where the probabilities of error are known, and no hypotheses must be made with regard to them as must be done in applying the method of least squares. The average risk is consequently equal to half the average error, because the positive errors or possible gains are not taken into account. This average error is different from the average square of error respecting which Gauss and Bessel have shown that it is a standard wherever the method of least squares can be applied.

The existing works upon Risk are as follows:—

Tetens, Einleitung zur Berechnung der Leibrenten und Anwartschaften (Introduction to the Calculation of Annuities and Reversions). Leipzig, 1786.

Raedell, Lebensfähigkeit von Versicherungs-Anstalten. Berlin, 1857. Pages 219-239.

Bremiker, Das Risico bei Lebensversicherungen (The risk attaching to the grant of insurances on lives). Berlin, 1859.

Zech, Das Risiko bei Lebensversicherungen. Tübingen, 1861.

Lachmund, Das Risiko bei Lebensversicherung (Elsner's Archiv für das Versicherungswesen. 1st Vol. Berlin, 1864. Pages 170-184).

Sprague, On the Limitation of Risks (Journal of the Institute of Actuaries. No. LXIII. April, 1866. Pages 20-39).

J. N. Tetens, Professor of Mathematics and Philosophy at Kiel, was the first to discuss the Theory of Risk. In his above-

mentioned excellent work, which, notwithstanding its antiquated way of dealing with the subject, is still well worth reading, Tetens enters upon a full discussion of the term Risk, and defines its meaning very clearly, without however extending it to the case of any given number of lives insured for different amounts. For the purpose of establishing his theory, the author puts in array a host of mathematical propositions, which my present limits will not allow me to notice. I will only remark that in my opinion modern writers do this deserving man an injustice, when they say that he attaches no fixed idea to the term Risk; for with him the uncertainty is not in his ideas, but in his method. From the greatest possible gain and the greatest possible loss he deduces an approximation to the risk for single annuities; and from these he passes to the risk for a greater number of lives. It will thus be readily understood that his theory must lead to most unsatisfactory results.

The subject of Risk appears to have been neglected for a period of 70 years, when Dr. Raedell brought forward a theory wherein, without giving any reasons for doing so, he transferred to the treatment of Risk certain propositions from the theory of average errors in the method of least squares. Starting from Tetens's definition, Raedell calculates directly for single annuities the average loss that should represent the risk run by the office in the *sale* of an annuity. With respect to the *purchase* of an annuity, he makes use of another method, which is erroneous, and brings out a result differing from the previous one. This difference, the author says, arises from the nature of things. In attempting to estimate the risk for a greater number of insurances, Raedell takes a fatal leap, and simply asserts "that the risk arising out of several contracts is equal to the square root of the sum of the squares of the risks arising out of the single contracts," adding an expression of regret that he was obliged to give this proposition without demonstration, as he had not succeeded in bringing it within the sphere of arithmetical reasoning.

The writers who come after Raedell, with the exception of the Englishman Sprague, likewise employ the method of least squares, treating the subject however very differently.

I propose to give, without further criticism, their principal views; and I will then give some reflections upon the method of least squares, from which it will appear that it is altogether inapplicable to the investigations connected with our subject.

Dr. Bremiker treats in the first place of the risk of single insurances, and identifies it with the square root of the sum of the

squares of all the errors divided by their number, while he forms the squares of the errors themselves by means of the deviations of all the single events from the average or most probable value. This definition therefore differs from Raedell's, but in passing on to the case of several insurances, he follows in the steps of his predecessor. However Dr. Bremiker justly censures the false distinction made by Raedell between purchase and sale.

Dr. Julius Zech, Professor of Mathematics and Astronomy in Tübingen, declares Dr. Bremiker's theory to be erroneous, because the deviations are not accidental but have been already provided for in calculating the premiums, and therefore the method of least squares is not applicable. Zech considers the risk to arise from the circumstance that in reality the rates of mortality and interest are different from those assumed in the calculation. Making certain hypotheses as to the deviations, Zech by means of the method of least squares finds the average error which, in his idea, must represent the risk in the case of single insurances. As to the risk attaching to several insurances, Zech says nothing.

Herr Julius Lachmund opens his treatise with an introduction that promises great things, and closes it with the wish that science may pronounce her verdict upon the construction of his formulæ; as to the principles on which they are based, no man can, in his opinion, entertain any doubt. For the "natural causes" of risk the author looks to the "periodical variations in the rate of mortality"; so that suppositions must be made as to the length of the period in which these fluctuations occur. "Here," says the author, "we may rely upon experience, and in doubtful circumstances, we shall do well not to go too deeply into these suppositions." Further on the author says: "moreover it is not necessary to be too anxious about these suppositions, &c.;" . . . . "and besides we have it in our power at the end of the next official year to compare the supposition with the actual result, to correct it accordingly, and to draw new conclusions from the basis of this new experience." He then gives some hypotheses, from which the average errors are found and then squared, &c. Then follow new hypotheses as to the proportions of the various periods, on which an experiment already begun by Herr Lachmund is to rest. Then he speaks of the periods in relation to different sums assured, with respect to which he is of opinion that theoretical and withal practically useful propositions will not be discovered without difficulty, while direct investigations will express the law with much greater facility. Then we have more opinions and a couple of examples, followed by the conclusion already mentioned.

Mr. T. B. Sprague, one of the Vice-Presidents of the Institute of Actuaries, gives a variety of propositions about the possible losses and their respective probabilities, without however coming to any conclusion. By "loss" he understands the full payment made by the Company, and demonstrates, among other things, that the total mathematical expectation of loss, as thus understood, is independent of the number of policies, a proposition however which is already well known. The writer however makes no claim to have solved the problem, for he says in conclusion "I have now only to remark that I am well aware that my subject is far from exhausted by the preceding remarks, and that much remains to be done to complete the mathematical part of the inquiry."

Gauss and Legendre, the inventors of the ingenious method of least squares, used it originally for calculating the orbits of planets and comets from observations, which, owing to the imperfection of our senses and the inaccuracy of our instruments, must always be subject to small errors even when made with the greatest precautions and skill. Subsequently its use was extended to all determinations of measure relating to physical science and geodesy. The general problem to be solved by this method, may be briefly enunciated as follows:

From n values of a known function, found by direct observation, for n different sets of values of the variables $u, v, w, \ldots\ldots,$ required to find the system of values of m constants involved in the function, m being less than n, which has, of all possible systems, the greatest probability.

Various analytical processes lead to the conclusion that we must so determine the constants that the sum of the squares of the errors which will occur under the supposition of these values, shall be a minimum. From this it follows that the function which expresses the probability of an error in terms of its magnitude will have the form

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx,$$

and will express the probability that the error lies within the infinitely small interval x and $x+dx$.

If we consider this function, we are struck with two of its principal properties. In the first place, it evidently presupposes the continuity of the errors; and in the second place, it gives equal probabilities for the positive and the negative errors of the same magnitude. Whence it follows that in the application of this method, if no further correction offers itself, these two con-

ditions must of necessity be fulfilled. Nevertheless we often see it applied to questions that are very far from satisfying these conditions. If I succeed in demonstrating the error of applying it to the determination of the probabilities of death, it will scarcely be necessary to show that it has nothing in common with the Theory of Risk.

I trust I may therefore be permitted to cite a very simple instance, in which two of the most celebrated writers on the subject of Vital Statistics, while agreeing completely as to the admissibility of the method of least squares, yet differ widely in respect of its application, whereas we cannot but think that in this case as with mathematics generally, no difference of opinion can possibly exist if the principles are well established.

In a paper published in Masius's *Rundschau der Versicherungen* (Third year, p. 336) Dr. Heym calculates the probability of death for a given age, from a series of contemporaneous registers of the numbers living and dying, according to the method of least squares. Putting

$$\begin{aligned} n_1 x &= m_1 \\ n_2 x &= m_2 \\ n_3 x &= m_3 \\ \vdots \quad & \quad \vdots \end{aligned}$$

he considers x as the quantity to be determined; $m_1, m_2, m_3, \ldots$ as the observed values of the function, the form of which is known. Consequently he obtains by means of well known rules

$$x = \frac{n_1 m_1 + n_2 m_2 + n_3 m_3 + \ldots .}{n_1^2 + n_2^2 + n_3^2 + \ldots .}$$

Dr. Fischer, in his *Elements of the Science of Life Assurance* (Grundzüge des auf menschliche Sterblichkeit gegründeten Versicherungswesens), Oppenheim a. R. 1860, (pp. 94-98) cannot approve of the fundamental form of equation adopted by Dr. Heym, for which he gives the sufficient reason that *à priori* there are no equal values to determine the amount of the errors whose sum is to be a minimum, inasmuch as they are derived from different numbers of living. Dr. Fischer first tries to determine the relative weight of these observations of different degrees of accuracy, by assuming it to be inversely as the number of the living; but the equations even then do not appear to him equally true, since, other circumstances being the same, the probability of dying will be more correctly derived from large numbers. The way out of this difficulty, the author thinks, is to assume that the weight of the

observations is inversely as the square roots of those numbers. I would now ask, if the whole operation is to depend on the humour of the calculator, to what end have any method at all? Is the method of least squares then not sufficiently established on its true foundation, to guide our steps in safety in its practical application? I should have thought that in the principles of that method as taught by Gauss in his *Theoria motus corporum cœlestium* we were in possession of a firm support to keep us from falling into error in applying it. It is remarkable that all the writers who apply it in connection with rates of mortality or with risk, have not a word to say by way of directing us where we are properly to look for the justification of this application.

We must not conclude however that we shall never, under certain conditions, be able to employ the method of least squares in future investigations into the law of mortality; but we must first have a rigorous demonstration of its applicability, for the mere management of it requires a clear and impartial judgment. We can indeed apply it in accordance with true principles to the example before given, obtaining the same result as we have previously arrived at by the direct investigation of the maximum value of the probability. Suppose we have k observations upon equal numbers of persons living

$$n_1 = n_2 = n_3 \;.\;.\;.\;.\;.\;. \; = n_k = n$$

giving different values of the probability of death, and that we consider the probability of death as the quantity which is subject to errors of observation; then the arithmetical mean of the observed probabilities of death will express the most probable value of the quantity sought, namely,

$$x = \frac{\frac{m_1}{n} + \frac{m_2}{n} + \frac{m_3}{n} + \ldots\ldots + \frac{m_k}{n}}{k}.$$

If however, n_1, n_2, n_3 . . . . n_k are of different magnitudes, then the weight of the observations will be proportionate to these quantities, which may be looked upon as representing the number of times the observations are repeated, and we shall therefore have by the help of well known rules:

$$x = \frac{n_1 \cdot \frac{m_1}{n_1} + n_2 \cdot \frac{m_2}{n_2} + \ldots\ldots + n_k \cdot \frac{m_k}{n_k}}{n_1 + n_2 + \ldots\ldots + n_k},$$

or simply $$x = \frac{m_1 + m_2 + \ldots\ldots\ldots + m_k}{n_1 + n_2 + \ldots\ldots\ldots + n_k}.$$

The reason why the method must be employed in the above manner, and why it leads to a true result, is not by any means simple, but on the contrary depends upon a subtle analysis that I will content myself with simply indicating. If we ask ourselves the question how great is the probability, that the actual probability of death lies between the limits $\frac{m}{n}+x$ and $\frac{m}{n}-x$, where $\frac{m}{n}$ is the observed probability of death, then in solving it we must express the required probability as a function of m and n. Supposing m and n to increase continually, then will this function approximate to the definite integral

$$\frac{h}{\sqrt{\pi}}\int_{-\delta}^{+\delta} e^{-h^2x^2}dx,$$

where δ denotes the special assumed value of x. We see that this function is exactly the same which, in the method of least squares, expresses the probability that the error lies within two given limits $+\delta$ and $-\delta$; only we must observe that the constant h, which in that method measures the precision, is here put for the expression

$$\sqrt{\frac{n^3}{2m(n-m)}}\,.$$

What we have said may suffice to show that the subjects we have been speaking of, must be treated with more profound investigations, and considerations of another kind than those hitherto employed. If we wish to carry out a mathematical investigation with success, we must enunciate the problem with precision, treat it in a clear and vigorous manner, and determine at each step the full meaning of our analytical processes. Deceptive analogies and a blind reliance on analytical identities will reduce our reasoning to a dull game of counters, where appearances are taken for reality.

Laplace, in the introduction to his master-work on the Theory of Probabilities, that wonderful monument of mathematical ingenuity, makes the following most appropriate remark: "The theory of probabilities, is virtually nothing but common sense reduced to figures. It enables us to define with precision what clear minds feel by a sort of instinct, often without their being able to explain how, &c."

With the Theory of Risk is connected a question of some practical importance, about which there has been much dispute, without any decided result being arrived at—we mean the well known question: how to calculate the Premium-Reserve for

insurances effected by annual premiums, when the circumstances connected with the management of the business require that the expenses should not be equally distributed over every year of the existence of the policies, but that more expense must be incurred in the first year or at the time of the insurance being effected, than in the following years.

That definition of the reserve which regards it from nearly the same point of view as we have done in the present paper, saying, that "it is formed from the balance of the premiums (improved at compound interest) which at the time of the valuation would not have been expended if the most probable case had constantly happened," leaves no room to doubt that the expenses already incurred, in so far as they are provided for in the premiums, can no more be included in that balance than are the claims which would have been paid under the supposition of the happening of the most probable case. The reserve so determined has this property that, together with the most probable income of the office and accumulation of interest, it exactly provides for the most probable claims and expenses; this being the necessary and sufficient condition of the reserve. In accordance with this principle we must, in making a valuation, on one side of the account set the present value of the future claims, and on the other side the present value of the future gross premiums, less the probable expenses of management. It is clear that the immediate expenses, as for example commuted commissions, inasmuch as they will not again appear among the probable future expenses although included in the gross premiums, may be entered on the credit side as future income, with the same propriety as the net premiums, or those premiums that are to provide for the payment of claims.

The circumstance, that the valuation in this case is not so simple as in the case of a uniform distribution of expenses, where we can use the net premiums on both sides, has given rise to differences of opinion. Some have tried to establish the contrary principle "that the assets and liabilities must both be capitalized by using only the net premiums," a rule which involves the conclusion that the office must consider the expenses incurred at the time of completing the insurances as entirely lost. If, however, we adopt this principle, we evidently disregard the real circumstances of the case, and take up the imaginary point of view of a uniform distribution of expenses; and the principle, if applied to other circumstances, is in contradiction to the true definition of Reserve.

What is the reason then that such views can have been taken up and maintained until now? Our answer is that the idea of Risk has not been strictly defined.

Without a true theory of Risk it was not easy to draw a distinct boundary line between the Premium-Reserve and the Capital-Reserve; the characteristic difference between which consists in the fact that the former forms an integral part of the liabilities; while the latter, as its name denotes, forms part of the capital, and is to serve as a guarantee fund against possible losses in consequence of deviations from the most probable event. It was seen that the payment of a commuted commission, by reducing the amount of the Premium-Reserve in the early years, while the liabilities of the office might become very large, caused a danger; and an endeavour was made to meet this danger by the introduction of a Reserve based on the net premiums, or a so-called Mortality-Reserve, in contradistinction to the necessary Balance-Reserve. This idea of an increased danger is in itself quite right; but, in looking for the means of meeting it, a false conclusion has been come to, from the fact being overlooked that the liabilities of the office will be increased on the one side just as much as the Premium-Reserve on the other, if the Mortality-Reserve is considered necessary for the fulfilment of the engagements of the office. If the office is to be always compelled to have this Reserve in hand, it will thereby be deprived of the means of paying the claims caused by a deviation from the average rate of mortality. But as the Balance-Reserve is sufficient for the average mortality, we have only to form a Capital-Reserve as a protection against possible deviations from the average, which is a question of "Risk," and herein lies the gordian knot to be untied.

Now it has been shown, that the average risk is so connected with the Premium-Reserve, that for each separate insurance the risk diminishes as the Premium-Reserve increases. Should the latter therefore under the given circumstances not increase rapidly enough at the first, the average risk will be so much the greater. The solution of the whole problem thus becomes simple: The greater the expenses in the earlier years, the larger must be the Capital-Reserve.

To look at the question from another point of view, let us suppose that we have a Table of Mortality which agrees with that of the Experience of the 17 English Offices, except that for the 25th year of life it shows a mortality greater by one per cent, and that we wish to calculate by this hypothetical table the uniform annual premium for an assurance of 100 thalers on a life of 24.

It is at once seen that this premium is greater than that found from the Experience Table, and that the present value of the difference is one thaler; because the single premium is higher by one thaler, and this excess has to be equally distributed over the remainder of life. The Mortality-Reserve at the expiration of the first year will evidently not include this thaler, as it is clearly provided for in the future net premiums.

If however this thaler is to pay, not claims, but commission, which is equally provided for in the calculation of the premiums, it is only a change in the kind of payment, which all the insured must bear in common; and this change can have no influence upon the reserve.

The greater expenditure of the first early years, be it for claims, commission, or anything else whatever, cannot affect the principle of the calculation of the Premium-Reserve, but only increase the risk and thus demand a larger Capital-Reserve. If we should look to the Premium-Reserve for a guarantee of the risk, we must of necessity arrive at the conclusion that the profits arising from a favourable mortality must not be divided, in the expectation that in future an unfavourable mortality will compensate for it, while we have already arbitrarily cut the thread that connects the past with the future. In point of fact, there are medical as well as mathematical reasons for this apprehension. If, however, the profits are nevertheless divided, it shows that we do not look to the Premium-Reserve for a guarantee against losses caused by a deviation from the average mortality; and we see withal of what importance to the security of the office a properly adjusted Capital-Reserve must be, for it is not a high Premium-Reserve but a Capital in proportion thereto that insures the stability of an insurance office.

In connection with the subject of commission by single payment, we must further observe that it must be confined within certain limits if the office would not *à priori* expose itself to losses. The principle is simply this: "The Balance-Reserve must never be negative, or what is the same thing, the office must never have to demand any contribution from the assured," for if an assured withdraws either by death or of his own free will while his reserve is negative, then the office loses an amount equal to that negative reserve. To such a loss, however, every office must be exposed which does not confine the payment of commission within the proper limits, however it may calculate its reserve.